Modular Invariants and Fischer-Griess Monster [1]

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Abstract. We show interesting relations between extremal partition functions of a family of conformal field theories and dimensions of the irreducible representations of the Fischer-Griess Monster sporadic group. We argue that these relations can be interpreted as an extension of Monster moonshine.

1 Introduction

Partition functions of conformal field theories with central charge $c = 24 \cdot k$, where $k = 1, 2, \ldots$, are modular invariants. There exists a relation between CFT with $c = 24$, i.e. $k = 1$, and the Monster sporadic group. This is the famous Monster moonshine theorem that states that coefficients of the q-expansion are related in a simple manner to the dimensions of the irreducible representations of the this largest sporadic group. Here, we are going conjecture a natural extension of this relation to the higher dimensional, i.e. $k > 1$, case. We are going to use lattices, or more precisely their Θ-functions to describe a given CFT. The q-expansion of a Θ-function of a lattice Λ is given as

$$Z_\Lambda = \sum_{x \in \Lambda} N(m) q^m \,, \tag{1}$$

where we sum over all vectors x, in the lattice Λ, with length $m = x \cdot x$. $N(m)$ is the number of vectors of norm m and $q \equiv e^{i\pi\tau}$ where τ is the modular parameter. The spectra of meromorphic conformal field theories can be expressed in terms of partition functions of even self-dual lattices. This means that the exponent m in the q-expansion will be necessarily an even number. Both self-duality and evenness of a lattice correspond to invariance of a partition function $\mathscr{Z}$ (which is closely related to Z_Λ) under the generators S and T of a modular group $SL(2, \mathbb{Z})$.

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Formally, Z_Λ of a d dimensional lattice Λ is a modular form of weight $d/2$. The partition function $\mathscr{Z}$ of a lattice is defined as follows

$$\mathscr{Z} = Z_\Lambda/\eta^{d/2}\,, \tag{2}$$

where $\eta(q) = q^{1/12}\prod_{m=1}^{\infty}(1-q^{2m})$ is Dedekind η-function which is a modular form of weight $1/2$. The partition functions of all the 24 dimensional even self-dual lattices (the Niemeier lattices) can be written as

$$\mathscr{Z} = [J + 24(h+1)]\,\eta^{24}\,, \tag{3}$$

where h is the Coxeter number of a given lattice. For example, $h = 0$ corresponds to the famous Leech lattice, $h = 30$ to the Niemeier lattice based on a root system of E_8^3, etc. Physically $24(h+1)$ corresponds to a number of massless states in a given theory. Using a technique presented in [1, 2], one can choose any Niemeier lattice Λ_1 to generate the Θ-function of another Niemeier lattice Λ_2. In [1] it was shown that it is possible to generate extremal partition functions [3] by taking the k^{th} power of (3) and treating $x_i = 24(h_i+1)$ (where $i = 1,\ldots,k$) as a free parameter.

2 CFTs with $c = 24\cdot k$: Systematic Approach

Different choices of constant parameters $x_i s$ correspond to different Θ-functions. One can find corresponding extremal partition functions, (2) and write them as q-expansions of the form

$$\prod_{i=1}^{k}(J+24+x_i) = \frac{1}{q^{2k}}\left[1 + \sum_{m=(k-1)}^{\infty} f_{2m}(x_1,\ldots,x_k)q^{2m-2k}\right] \tag{4}$$

As an example, we will work with a choice of $k-1$ parameters x_is that eliminates coefficients of all but one term with negative powers in the q-expansion that correspond (in a field theoretic language) to tachyonic states. In this setup we are left with only one free parameter x_k. Different choices of x_k would correspond to different partition function candidates for conformal field theories with $c = 24\cdot k$. Here we list the first three cases:

$$\mathscr{G}_1(x_1) = \frac{1}{q^2} + (24 + x_1) + 196884q^2 + \ldots \tag{5a}$$

$$\mathscr{G}_2(x_2) = \frac{1}{q^4} + (393192 - 48x_2 - x_2^2) + 42987520q^2 + \ldots \tag{5b}$$

$$\mathscr{G}_3(x_3) = \frac{1}{q^6} + (50319456 - 588924x_3 + 72x_3^2 + x_3^3) + 2592899910q^2 + \ldots \tag{5c}$$

Notice that in each case all of the tachyonic states (except the lowest one) are absent. Since the allowed values [4] of the coefficient of q^0 are integers that run from zero to the value of the q^2 coefficient, one can easily find the number of "allowed" partition functions in $24k$ dimensions [1].

3 Monster Moonshine and its Extension

The extremal 24 dimensional case has been shown to be related to the Fischer-Griess monster group. In mathematics this fact is known as Monster moonshine ([5] and [6]). One can evaluate $\mathscr{G}_1$ at $x_1 = -24$ which corresponds to the j-invariant to find

$$j = \frac{1}{q^2} + 196884q^2 + 21493760q^4 + 864299970q^6 + 20245856256q^8 + \dots . \quad (6)$$

The coefficients of this expansion decompose into dimensions of the irreducible representations of the Monster[2], where we use the notation $j = \frac{1}{q^2} + j_2 q^2 + j_4 q^4 + \dots$. Following this interpretation of the Monster, one can easily generate moonshine in higher dimensional cases, i.e., one can express coefficients of any partition functions, for example $\mathscr{G}_k(x_k)$, for any choice of k, in terms of the dimensions of irreducible representations of the Monster group. We present a few of our results [1] in Table-1, where coefficients of $G_k(x_k)$ are expressed in terms of the coefficients of the invariant function j, that (via the original Monster moonshine) are related to the Monster. We notice [1] that the coefficients g_{2n} fall into patterns with period $k!$, and conjecture that this periodicity also continues to hold for all k. The polynomial conditions to be satisfied to find the extremal partition functions for large k become increasingly more difficult to solve with increasing k, so we do not have results for $k > 6$.
Table-1 give the general periodicity in coefficients of $\mathscr{G}_k(x_k)$ for $k < 6$. To summarize, when $k = 1$ it is known via standard Monster Moonshine that the coefficients of j decompose into Monster representations [6]. The fact that all the higher k coefficients also decompose into Monster representations indicates that they have large symmetries containing the Monster and the fact that they have these symmetries may indicate that they are related to $24k$ dimensional lattices.

[2] for explicit realization of the Monster moonshine see [1].

$k=2$	$k=2$	$k=3$	$k=3$
g_{4i+2}	$2j_{2(4i+2)}$	g_{6i+2}	$3j_{3(6i+2)}$
g_{4i+4}	$2j_{2(4i+4)}+j_{2(2i+2)}$	g_{6i+4}	$3j_{3(6i+4)}$
		g_{6i+6}	$3j_{3(6i+6)}+j_{2i+2}$
$k=4$	$k=4$	$k=5$	$k=5$
g_{8i+2}	$4j_{4(8i+2)}$	g_{10i+2}	$5j_{5(10i+2)}$
g_{8i+4}	$4j_{4(8i+4)}+2j_{2(2i+4)}$	g_{10i+4}	$5j_{5(10i+4)}$
g_{8i+6}	$4j_{4(8i+6)}$	g_{10i+6}	$5j_{5(10i+6)}$
g_{8i+8}	$4j_{4(8i+8)}+2j_{(8i+8)}+j_{2i+2}$	g_{10i+8}	$5j_{5(10i+8)}$
		g_{10i+10}	$5j_{5(10i+10)}+j_{2i+2}$

Table 1. Periodicity of the coefficients g_n for $c = 24 \cdot k$ extremal partition functions $\mathscr{G}_k$ in terms of coefficients j_{2n} of the modular function j.

4 Final Comments

We have shown that there exists a family of transformations, more general then of simple $\mathbb{Z}_2$ form, that relates the members of the class of holomorphic conformal field theories, i.e., the Niemeier lattices. Furthermore these transformations connect non-Niemeier modular invariant $c = 24$ Θ-functions. These results generalize to any $c = 24 \cdot k$ case (in particular, when the resulting parametrization corresponds to $24k$ dimensional lattice). In 24 dimensions there is one extremal partition function, corresponding to a Leech lattice, whose q-expansion coefficients can be written in the form of a linear combinations of dimensions of the irreducible representations of the Monster group; this relationship is at the core of the Monster Moonshine. The decomposition is possible only in the extremal case, since in all other cases a non-zero constant term in the q-expansion would be present. The presence of this term in the expansion would imply the introduction of an enormous unnatural set of singlets in the decomposition. We believe a similar situation occurs at $24k$ where we have extended this argument. Instead of a Leech lattice we have to deal with higher dimensional extremal partition functions (but note [1] that there is more than one type of extremal partition function in 24k dimensions). Existence of $24k$ extremal lattices for $k > 2$ is only a conjecture [3], but our results are consistent with and provides supporting evidence for this conjecture. One can use our generalized version of Monster Moonshine to postulate that we already have the q-expansion of higher dimensional extremal lattices, and that the symmetry provided by the Monster decomposition can be used to learn more about these lattices.

We suggest that the extremal $\mathscr{G}$ partition functions will generate new $c = 24 \cdot k$ CFTs. We know the first at $k = 1$ corresponds to the Leech lattice, the second $k = 2$ case also corresponds to the known lattice, P_{48}, and since the higher k cases all possess Monster symmetry we conjecture that it is likely that they correspond to CFTs constructed on extremal lattices in $24k$-dimensions. To the best of our knowledge, Monster symmetry was not known to come into play except at $k = 1$.

Using the techniques presented in [1], one can construct a large class of conformal field theories with central charge that is a multiple of 24. We have demonstrated (or at least conjectured) the possibility of a new realization of Monster moonshine. This is realized as a periodicity in a pattern of coefficients in q-expansions of the extremal partition functions.

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